AD-A160 214 STRUCTURAL PROPERTIES OF RANDOMIZED TIMES(U) JOHNS HOPKINS UNIV BALTIMORE MD DEPT OF MATHEMATICAL SCIENCES A F KARR ET AL. DEC 84 TR-421 AFOSR-TR-85-0814 1/1

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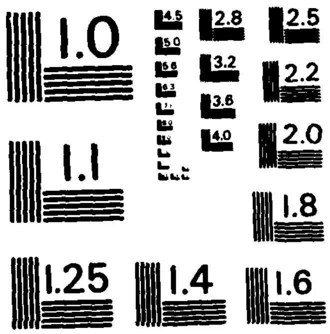
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REPORT DOCUMENTATION PAGE

1a. REPORT SECURITY CLASSIFICATION	1b. RESTRICTIVE MARKINGS
2a. SECURITY CLASSIFICATION AUTHORITY: UNCLASSIFIED	3. DISTRIBUTION/AVAILABILITY OF REPORT: Approved for public release; distribution unlimited.
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE: ----------	
4. PERFORMING ORGANIZATION REPORT NUMBER(S): N/A	5. MONITORING ORGANIZATION REPORT NUMBER(S): AFOSR-TR- 85-0814
6a. NAME OF PERFORMING ORGANIZATION: Johns Hopkins University; 6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION: AFOSR
6c. ADDRESS (City, State and ZIP Code): Baltimore, MD 21218	7b. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, D.C. 20332-6448
8a. NAME OF FUNDING/SPONSORING ORGANIZATION: AFOSR; 8b. OFFICE SYMBOL (If applicable): NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER: AFOSR-82-0029
8c. ADDRESS (City, State and ZIP Code): Bldg. 410, Bolling AFB, D.C. 20332-6448	10. SOURCE OF FUNDING NOS.: PROGRAM ELEMENT NO. 61102F; PROJECT NO. 2304; TASK NO. A5; WORK UNIT NO.
11. TITLE (Include Security Classification): Structural properties of randomized times	
12. PERSONAL AUTHOR(S): Alan F. Karr and A.O. Pittenger	
13a. TYPE OF REPORT: Interim; 13b. TIME COVERED: FROM ____ TO ____	14. DATE OF REPORT (Yr., Mo., Day): December 1984; 15. PAGE COUNT: 19
16. SUPPLEMENTARY NOTATION	
17. COSATI CODES: FIELD; GROUP: XXXXXX; SUB. GR.	18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number): Randomized optional times, randomized quasi-terminal times, randomized terminal times, remplissage by hitting times

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

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DTIC ELECTE OCT 15 1985 S A

20. DISTRIBUTION/AVAILABILITY OF ABSTRACT: UNCLASSIFIED/UNLIMITED X; SAME AS RPT. X; DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION: Unclassified
22a. NAME OF RESPONSIBLE INDIVIDUAL: Brian W. Woodruff, MAJ, USAF	22b. TELEPHONE NUMBER (Include Area Code): (202)767-5027; 22c. OFFICE SYMBOL: NM

DD FORM 1473, 83 APR — EDITION OF 1 JAN 73 IS OBSOLETE.

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE

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Structural Properties of Randomized Times

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Technical Report 421
December, 1984

* Research supported in part by Air Force Office of Scientific Research grant AFOSR 82-0029.

** Research supported in part by NSF grant MCS-8404539.

Abstract

Suppose a measure μ dominates a measure η in the ordering induced by the excessive functions of a transient Markov process. Rost shows than η can be represented as the distribution of the process stopped at a randomized optional time and started with initial distribution μ. In this paper we introduce the shift operator to the class of randomized optional times, inducing the class of randomized quasi-terminal times and that of randomized terminal times. We analyze the algebraic properties of these classes and obtain some compactness results for the class of randomized quasi-terminal times. Some applications, including remplissage by hitting times, are presented.

1. Introduction. Suppose we have a Markov process X on a suitable state space S with Borel σ - algebra $\mathcal{S}$. If A is a measurable subset of S and μ an initial distribution for the process, then the μ - hitting distribution of A is defined by starting the process with distribution μ and running it until it hits A for the first time. This probabilistic construct has close links with the balayage of measures, a potential - theoretic construct, and plays a central role in probabilistic potential theory. (The geometric meaning of these concepts becomes clearer when A is the boundary of an open connected set G and μ is required to have support in G.)

In earlier work ([7], [8], [9]) the authors investigated the reverse of this construction, the so-called inverse balayage problem. One is given a distribution η on A and tries to characterize the family of measures whose balayage onto A is η. For finite state Markov chains rather straightforward techniques yield a complete description of the convex set of measures which balayage onto the given η. The same kind of analysis works for a diffusion on an interval, but more sophisticated techniques are required for general state spaces.

The questions arising in the inverse balayage problem are closely connected with Skorokhod's problem and work of Rost. Rost [12] shows that if μ dominates η in a partial ordering defined by the excessive functions for X, then η can be realized as the distribution of the process started with initial distribution μ and stopped using a randomized optional time $\{T_u : 0 < u \leq 1\}$:

$$\eta(dx) = \int_0^1 P^{\mu}\big(X(T_u) \in dx\big)du. \tag{1.1}$$

One interpretation of (1.1) is that η is represented as a sort of convex combination of optional times, and this motivates the definition of a convexity structu e on the space of randomized optional times. The inverse balayage problem can be viewed as a search for measures μ which satisfy (1.1) with $T_u = T_A$ for all $0 < u \leq 1$.

Another related result is that of Heath [6]. Drawing heavily on potential-theoretic work of Mokobodski and Watanabe [16], Heath shows that for discrete-time processes the randomized times in (1.1) can be chosen to be "nested" terminal times: for each u, T_u is a terminal time and $u < v$ implies $T_u \leq T_v$ a.s. If one specializes to finite state Markov chains, it is easy to obtain that result using a less sophisticated approach - we call it remplissage via hitting times and present details in Section 5 as an example of the construction of a randomized terminal time.

In [1] Baxter and Chacon studied compactness properties of randomized times in a suitable topology, and Falkner [3] remarked on the use of this compactness to obtain (1.1). (Several of Falkner's related papers are listed in the references.)

Motivated by all of these ideas and results, we examine here structural properties of randomized times in the presence of the shift operator. Of special interest is the class of randomized quasi-terminal times, which lies between randomized optional times and randomized terminal times. This class is motivated by the interaction between the shift operator and basic algebraic operations, and the requisite definitions are given in Section 2.

In Section 3 we discuss algebraic consequences of the definitions, while Section 4 is devoted to sequential compactness of quasi-terminal times. Technical questions of null sets arise and a complete generalization of Baxter and Chacon's results to quasi-terminal times does not seem possible without additional hypotheses. In Section 5 we illustrate some of these ideas, including applications of sequential compactness results to Markov chains and to Rost's original proof of (1.1) for Markov processes.

2. Randomized times and random measures. The motivation for this work comes from Markov processes, and we record some of the basic definitions. Let $\mathcal{S}$ denote the Borel σ - algebra on a state space S assumed to be Lusin - that is, S is homeomorphic to a Borel subset of a compact topological space. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a Markov process on S. We assume that the associated semigroup $P_t f(x) = P^x[f(X_t)]$ maps Borel functions into Borel functions and that the process is right-continuous. (We follow the convention of using P^x to denote both a probability and an expectation.)

Our immediate concern is less the Markov process itself than the filtration $(\mathcal{F}_t : t \geq 0)$, which we assume to be right-continuous and the usual completion of the separable σ - algebra

$$\mathcal{F}_t^0 = \sigma\{X_s : s \leq t\}. \tag{2.1}$$

One way of introducing the randomization is via the product space

$$\tilde{\Omega} = \Omega \times [0, \infty] \;,\; \tilde{\mathcal{F}}_t = \mathcal{F}_t \times \mathcal{B}, \tag{2.2}$$

where $\mathcal{B}$ denotes the Borel sets on $[0, \infty]$. The first class of randomized times with which we are concerned is given, for example, in Baxter and Chacon [1]; we use a slight variation of their definition.

(2.3) Definition. $\mathcal{T}_0$ is the set of mappings $T : \tilde{\Omega} \to [0, \infty]$ with the properties

(a) $T(., u)$ is optional for each u - that is,

$\{\omega: T(\omega,u) \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$.

(b) $T(\omega,.)$ is nondecreasing and left-continuous for each ω.

(We define $T(\omega, 0) = T(\omega, 0^+)$.)

(c) $u_T(\omega) \equiv \inf \{u : T(\omega, u) = \infty\} < \infty$.

The set of random measures corresponding to $\mathcal{T}_0$ coincides with the following class.

(2.4) Definition. $\mathcal{A}_0$ is the set of random measures A on $[0, \infty]$ with the properties

(a) $A(\cdot, s) \in \mathcal{F}_s$, all $s \in [0, \infty]$, where $A(\omega, s) = A(\omega, [0, s])$.

(b) $A(\omega, s)$ is right-continuous in s.

(c) $A(\omega, [0, \infty)) < \infty$.

The relationship between $\mathcal{T}_0$ and $\mathcal{A}_0$ is defined in the usual manner: given T in $\mathcal{T}_0$ define A_T by

(2.5) $$A_T(\omega, s) = \sup\{u : T(\omega, u) \leq s\}.$$

Then A_T is nondecreasing, and it is easy to check that

(2.6) $$\{\omega : A_T(\omega, s) \geq u\} = \{\omega : T(\omega, u) \leq s\}.$$

Routine computations show that $A_T \in \mathcal{A}_0$. In fact, the mapping $M_1 : \mathcal{T}_0 \to \mathcal{A}_0$ defined by

(2.7) $$M_1(T) = A_T$$

has the following properties.

(2.8) Lemma. The mapping M_1 is one-to-one and onto, and its inverse map $M_2 : \mathcal{A}_0 \to \mathcal{T}_0$ is given by $M_2(A) = T_A$, where

(2.9) $$T_A(\omega, u) = \inf\{s : A(\omega, s) \geq u\}.$$

We now bring in shift operators using the "big shift". For $T \in \mathcal{T}_0$ and $s \geq 0$ define

(2.10a) $$\Theta_s T(\omega, u) = s + T(\theta_s \omega, u);$$

(2.10b) $$\Theta_s A(\omega, t) = A(\theta_s \omega, t-s) 1(s \leq t \leq \infty).$$

The definition of Θ_s on random measures seems to have been introduced by Sharpe (see [2] for example), and $\Theta_s T$ appears in one guise or another throughout the literature, usually in the simpler form of T_s. (Note that we should really define two shifts, since one applies to random times and the other to random measures.)

Next we define subclasses of $\mathcal{A}_0$ and $\mathcal{T}_0$ suggested by the shift operator and the applications mentioned in the introduction.

(2.11) Definition. Let

$\mathcal{T}_1 = \{T \in \mathcal{T}_0$: for all $0 \leq s < \infty$, $\Theta_s T \in \mathcal{T}_0$ and for all $0 \leq r \leq s$ and $0 \leq u$, $\Theta_r T(\omega, u) \leq \Theta_s T(\omega, u)\}$;

$\mathcal{A}_1 = \{A \in \mathcal{A}_0$: for all $0 \leq s < \infty$, $\Theta_s A \in \mathcal{A}_0$ and for all $0 \leq r \leq s$, $0 \leq t$, $\Theta_r A(\omega, t) \geq \Theta_s A(\omega, t)\}$;

$\mathcal{T}_2 = \{T \in \mathcal{T}_1$: for all $0 \leq s < \infty$, $s < T(\omega, u)$ implies $\Theta_s T(\omega, u) = T(\omega, u)\}$;

$\mathcal{A}_2 = \{A \in \mathcal{A}_1 : A(\omega, s) < A(\omega, t)$ implies $\Theta_s A(\omega, t) = A(\omega, t)\}$.

These definitions can be weakened by allowing inequalities to fail on null sets depending on the time parameters. Thus $\tilde{A}_1$ denotes the set of $A \in A_0$ such that $\Theta_s A \in A_0$ for all $s \geq 0$ and such that for $0 \leq r \leq s$ there is a null set depending on r and s off which $\Theta_r A(\omega, t) \geq \Theta_s A(\omega, t)$ for all t. Similarly, $\tilde{T}_1$ shall denote the analogous class of randomized times. The need for these classes arises in Section 5, where we apply our results to the representation theorem of Rost [12], and only the sets $\tilde{A}_1$ and $\tilde{T}_1$ are available.

A random time in T_2 has the property that $T(\cdot, u)$ is a terminal time for all u. One can construct such a time by defining $T(\cdot, u)$ as the hitting time of a Borel set B_u, where $0 < u < v \leq 1$ implies $B_u \supset B_v$. (An example of how such times arise in practice appears in Section 5.) Times in T_2 shall be referred to as <u>randomized terminal times</u>, while those in the *intermediate class* T_1 *are called* <u>randomized quasi-terminal times</u>, since the defining property comprises part of the definition of a terminal time.

We single out those times in T_i which correspond to ordinary random times.

(2.12) <u>Definition</u>. A time T in T_0 is called <u>natural</u> if $T(\omega, u) = T(\omega)$ for all $u \leq u_T$. Correspondingly, we call a natural T a <u>quasi-terminal time</u> or a <u>terminal time</u> if it is also in T_1 or T_2, respectively.

We conclude these definitions by showing T_i and A_i are related in the same way that T_0 and A_0 are related and also that the shift operaters commute with M_1 and M_2. We leave it to the reader to check that the same results apply to $\tilde{T}_1$ and $\tilde{A}_1$.

(2.13) <u>Proposition</u>. For i = 1, 2, M_1 and M_2 restricted to T_i and A_i respectively are inverses. Furthermore, for all $s \geq 0$

(2.14) $$\Theta_s M_2 = M_2 \Theta_s \quad , \quad \Theta_s M_1 = M_1 \Theta_s .$$

Proof: Suppose $T = M_2(A)$ with A in $\mathcal{A}_1$. Then with $u > 0$ and $r \le s$

(2.15)
$$\begin{aligned}
\Theta_s T(\omega, u) &= s + T(\theta_s \omega, u) \\
&= s + \inf\{t : A(\theta_s \omega, t) \ge u\} \\
&= s + \inf\{t : \Theta_s A(\omega, s+t) \ge u\} \\
&\ge s + \inf\{t : \Theta_r A(\omega, s+t) \ge u\} \\
&= r + \inf\{t \ge s-r : A(\Theta_r \omega, t) \ge u\} \\
&\ge \Theta_r T(\omega, u).
\end{aligned}$$

Line three above translates to $\Theta_s M_2(A) = M_2(\Theta_s A)$, and the entire sequence gives T in $\mathcal{T}_1$. Now set $r=0$. If $t_0 = T(\omega, u) > s$, then for $s+t_1 < t_0 \le s+t_2$, we have

$$A(\omega, s+t_1) < u \le A(\omega, s+t_2).$$

Hence, if $A \in \mathcal{A}_2$, $A(\omega, s+t_2) = \Theta_s A(\omega, s+t_2)$ and there is equality in line four of (2.15). Further, the infimum implicit in the last line is over $t > s$, so that T must be in $\mathcal{T}_2$.

For the other direction we have an analogous argument. Suppose $r < s < t$ and $T \in \mathcal{T}_1$. Then with $A = M_1(T)$,

$$\begin{aligned}
\Theta_s A(\omega, t) &= A(\theta_s \omega, t-s) \\
&= \sup\{u : T(\theta_s \omega, u) \le t-s\} \\
&= \sup\{u : \Theta_s T(\omega, u) \le t\} \\
&\le \sup\{u : T(\omega, u) \le t\} \\
&= \sup\{u : T(\Theta_r \omega, u) \le t-r\} = \Theta_r A(\omega, t),
\end{aligned}$$

confirming $\Theta_s M_1(T) = M_1(\Theta_s T)$ and A in $\mathcal{A}_1$. If in addition $T \in \mathcal{T}_2$ and $A(\omega, s) = u_0 < u_1 = A(\omega, t)$, then $T(\omega, u) > s$ for $u > u_0$, which gives

$\Theta_s T(\omega, u)$ equal to $T(\omega, u)$. If r is set equal to zero it follows that the suprema above can be restricted to $u \geq u_0$ and that we have equality in the fourth line, completing the proof. $\square$

3. Algebraic structure on $\mathcal{A}_i$ and $\mathcal{T}_i$. Suppose X is two-dimensional Brownian motion absorbed at the circle of radius two. If μ denotes the point mass at the origin and η a probability measure with half of its mass at the origin and half uniformly distributed on the unit circle, then μ dominates η in the ordering induced by the excessive functions. Furthermore, η can be realized as the process with initial distribution μ stopped "half the time" at zero and otherwise at the unit circle. If D_0 is the hitting time of (0, 0) and D_1 is the hitting time of the unit circle, then one could think of η as the process with initial distribution μ stopped at "$\frac{1}{2} D_0 + \frac{1}{2} D_1$".

We make this approach rigorous by defining an appropriate algebraic structure on $\mathcal{T}_0$ using $\mathcal{A}_0$ and the mapping M_2. Before proceeding let us note that the intuition behind these ideas is not new; Meyer, for example, alludes to order and convexity properties in his discussion [11] of Baxter and Chacon's work.

Addition and positive scalar multiplication on $\mathcal{A}_0$ are defined in terms of the distribution functions.

(3.1) Definition. For $A_1, A_2 \in \mathcal{A}_0$ and $c \geq 0$ let

$$(A_1 + A_2)(\omega, t) = A_1(\omega, t) + A_2(\omega, t);$$

$$(cA_1)(\omega, t) = c \cdot A_1(\omega, t).$$

Since the properties characterizing $\mathcal{A}_0$ are satisfied for both $A_1 + A_2$ and cA_1, $\mathcal{A}_0$ is closed under these operations. It is easy to check that $\mathcal{A}_1$ is also closed under both operations and that $\mathcal{A}_2$ is closed under scalar multiplication. (The latter assertion follows from the use of $c \cdot \Theta_s A = \Theta_s(cA)$.)

However, $\mathcal{A}_2$ need not be closed under addition. For example if

$$A_i(\omega, t) = 1(T_i(\omega) \leq t \leq \infty) \ , \ i = 1,2,$$

where the T_i are finite terminal times, then the inequality $A_i(\omega, s) < A_i(\omega, t)$ obviously implies equality of $\Theta_s A_i(\omega, t)$ and $A_i(\omega, t)$. However, if

$$T_1(\omega) < s < T_2(\omega) < t < \Theta_s T_1(\omega),$$

then $(A_1 + A_2)(\omega, s) < (A_1 + A_2)(\omega, t)$, but

$$\Theta_s(A_1 + A_2)(\omega, t) = 1 < 2 = (A_1 + A_2)(\omega, t).$$

The following result summarizes these observations.

(3.2) Lemma. *The sets $\mathcal{A}_0$ and $\mathcal{A}_1$ are positive cones under addition and positive scalar multiplication, while $\mathcal{A}_2$ is closed under positive scalar multiplication. Furthermore, if algebraic operations on $\mathcal{T}_i$ are defined by*

$$T_1 + T_2 \equiv M_2(A_1 + A_2)$$

and

$$c \cdot T_1 = M_2(cA_1),$$

then $\mathcal{T}_0$ and $\mathcal{T}_1$ are positive cones, and $\mathcal{T}_2$ is closed under scalar multiplication.

Proof. The definitions of multiplication and addition on $\mathcal{A}_0$ show these operations satisfy the required algebraic properties. Moreover, these properties are preserved under the bijection M_2, so that $\mathcal{T}_0$ is a positive cone; for example,

$$(a \cdot T_0) + (a \cdot T_1) = M_2(aA_0 + aA_1) = a \cdot (T_0 + T_1).$$

Since M_2 maps $\mathcal{A}_1$ onto $\mathcal{T}_1$, we easily deduce that $\mathcal{T}_1$ is a positive cone. Note that part of the verification uses commutativity of scalar multiplication with the big shift:

$$\Theta_s(cT) = \Theta_s\big(M_2(cA)\big) = M_2\big(\Theta_s(cA)\big)$$

$$= M_2(c\Theta_s A) = cM_2(\Theta_s A) = c \cdot \Theta_s A.$$

We omit the remaining details. $\square$

The effect on T of scalar multiplication is to rescale the randomizing parameter:

$$cT(\omega, u) = \inf\{t : cA(\omega, t) \geq u\}$$

$$= \inf\{t : A(\omega, t) \geq u/c\} = T(\omega, u/c).$$

Addition has the effect of mixing up the values assumed by the constituent times, an effect noted by Meyer [10]. Thus, for the example preceding (3.2)

$$(T_1 + T_2)(\omega, u) = \inf\{r : A_1(\omega, r) + A_2(\omega, r) \geq u\}$$

$$= \begin{cases} T_1(\omega), & 0 < u \leq 1 \\ T_1(\omega) + T_2(\omega), & 1 < u \leq 2 \\ \infty, & 2 < u. \end{cases}$$

Order and convexity structures are defined in the obvious way.

(3.3) Definition. $A_1 \leq A_2$ iff $A_1(\omega, t) \leq A_2(\omega, t)$ for all $t \geq 0$; $T_1 \leq T_2$ iff $T_1(\omega, u) \leq T_2(\omega, u)$ for all $u \geq 0$.

To ease the exposition we assume we are working on a fixed Ω with the inequalities holding for all ω. However, the presentation can be easily modified so that the inequalities are valid except on a null set.

(3.4) Definition. For A_1 , A_2 in $\mathcal{A}_0$, let

$$(A_1 \vee A_2)(\omega, t) \equiv A_1(\omega, t) \vee A_2(\omega, t);$$

$$(A_1 \wedge A_2)(\omega, t) \equiv A_1(\omega, t) \wedge A_2(\omega, t).$$

Similarly, given T_1 , T_2 in $\mathcal{T}_0$ let

$$(T_1 \overset{\wedge}{\vee} T_2)(\omega, u) \equiv (T_1(\omega, u)) \overset{\wedge}{\vee} \big(T_2(\omega, u)\big).$$

The sets $\mathcal{A}_i$ and $\mathcal{T}_i$ have a variety of lattice properties, but we will content ourselves with stating only a few explicitly.

(3.5) Lemma. The sets $\mathcal{A}_i$ and $\mathcal{T}_i$, i = 1 and 2, are closed under $\wedge$ and $\vee$. In addition scalar multiplication and the shift operators commute with $\wedge$ and $\vee$, while

$$(3.6)\qquad \begin{cases} A_1 + A_2 = (A_1 \vee A_2) + (A_1 \wedge A_2) \\ T_1 + T_2 = (T_1 \wedge T_2) + T_1 \vee T_2). \end{cases}$$

Finally,

$$(3.7)\qquad \begin{cases} M_2(A_1 \overset{\vee}{\wedge} A_2) = M_2(A_1) \overset{\wedge}{\vee} M_2(A_2) \\ M_1(T_1 \overset{\vee}{\wedge} T_2) = M_1(T_1) \overset{\wedge}{\vee} M_1(T_2). \end{cases}$$

Proof. It is easy to check that $\mathcal{A}_0$ is closed under $\vee$ and $\wedge$ and that the first part of (3.6) holds. Moreover, since Θ_s commutes with both $\wedge$ and $\vee$, the same assertions hold for $\mathcal{A}_1$.

Now if $T = M_2(A_1 \vee A_2)$, then

$$\begin{aligned} T(\omega, u) &= \inf\{t : A_1(\omega, t) \vee A_2(\omega, t) \geq u\} \\ &= \inf\{t : A_1(\omega, t) \geq u\} \wedge \inf\{t : A_2(\omega, t) \geq u\} \\ &= T_1(\omega, u) \wedge T_2(\omega, u). \end{aligned}$$

Similarly $M_2(A_1 \wedge A_2) = M_2(A_1) \vee M_2(A_2)$. This shows $\mathcal{T}_0$ and $\mathcal{T}_1$ are closed under $\wedge$ and $\vee$ and also that the equations in (3.7) hold. Finally, the second part of (3.6) follows from (3.7) and the first part of (3.6). □

The example preceding (3.2) shows that $\mathcal{T}_2$ need not be closed under $\vee$. However we do have a partial result.

(3.8) Lemma. $\mathcal{A}_2$ and $\mathcal{T}_2$ are closed under $\vee$ and $\wedge$ respectively.

Proof. Let A_1 and A_2 be in $\mathcal{A}_2$. Suppose $s < t$ and

(3.9) $$(A_1 \vee A_2)(\omega, t) > (A_1 \vee A_2)(\omega, s).$$

If $A_i(\omega, t) > A_i(\omega, s)$ for $i = 1, 2$, then

(3.10) $$\Theta_s(A_1 \vee A_2)(\omega, t) = (\Theta_s A_1 \vee \Theta_s A_2)(\omega, t)$$
$$= \Theta_s A_1(\omega, t) \vee \Theta_s A_2(\omega, t) = (A_1 \vee A_2)(\omega, t).$$

If $A_2(\omega, s) = A_2(\omega, t)$, necessarily $A_1(\omega, t) > A_2(\omega, t)$ and $A_1(\omega, t) > A_1(\omega, s)$ for (3.9) to hold. This suffices for (3.10) as well, and the assertion for $\mathcal{A}_2$ follows. The assertion for $\mathcal{T}_2$ then follows using the usual mapping. $\square$

It is natural to ask about convexity properties, which requires that we use bases of the positive cones described above. In doing this we impose a condition for all ω - or again for a full set if we use the weaker definitions mentioned above.

(3.11) Definition. For $i = 0, 1, 2$, let

$$\mathcal{T}_i(1) = \mathcal{T}_i \cap \{T : u_T(\omega) \leq 1\};$$

$$\mathcal{A}_i(1) = \mathcal{A}_i \cap \{A : A(\omega, [0, \infty)) \leq 1\}.$$

Further, we require natural times in this context to have $u_T(\omega) = 1$. It is easy to check that M_1 and M_2, restricted to $\mathcal{T}_i(1)$ and $\mathcal{A}_i(1)$, remain inverses of one another. The sets defined in (3.11) are those used by Baxter and Chacon. [1].

The next result identifies extreme points.

(3.12) Theorem. For $i = 0$ and $i = 1$, $\mathcal{A}_i(1)$ and $\mathcal{T}_i(1)$ are convex sets. The extreme points of $\mathcal{T}_0(1)$ correspond to natural times, while those of $\mathcal{T}_1(1)$ are quasi-terminal times.

Proof. The first assertion is immediate from the definitions. For the second, given $a \in (0, 1)$ and $A \in \mathcal{A}_i(1)$, define $A_1 = a^{-1}(a \wedge A_1)$ and $A_2 = \max\{0, (1 - a)^{-1}(A - a)\}$. It is easy to check that A_1 and A_2 are also in $\mathcal{A}_i(1)$ and that $A = aA_1 + (1 - a)A_2$. If A is an extreme point, $A = A_1 = A_2$. Thus if $A(\omega, t) < a$, $A_2(\omega, t) = 0 = A(\omega, t)$. If $A(\omega, t) > a$, $A_1(\omega, t) = 1 = A(\omega, t)$. Hence for each ω there is a $T_0(\omega)$ such that $A(\omega, t) = 1(T_0(\omega) \leq t \leq \infty)$. (Note that $T_0 = \infty$ is a possibility and also that we could have done the foregoing analysis on a set of probability one.) It is easy to check that if

$$T(\omega, u) = \begin{cases} T_0(\omega) & 0 \leq u \leq 1 \\ \infty & u > 1, \end{cases}$$

then T is a natural time if $A \in \mathcal{A}_0(1)$ and is a quasi-terminal time if $A \in \mathcal{A}_i(1)$. The proof that natural and natural quasi-terminal times are extreme points is trivial, and details are omitted. $\square$

Having characterized extreme points we proceed one step further and characterize edges, i.e., faces of dimension one.

(3.13) Proposition. Let T_1 and T_2 be extreme points of $\mathcal{T}_0(1)$. Then T_1 and T_2 define on edge iff one of the times is smaller than the other, say $T_1 \leq T_2$, and $\mathcal{F}(T_1)$ is trivial.

Remark. If we are working with respect to a fixed measure P^η, then the null sets should be interpreted with respect to that measure. Indeed, the proposition implies that edges will exist only for initial measures whose support is at one point.

Proof. Suppose T_1 and T_2 are natural, $T_1 \leq T_2$ and $\mathcal{F}(T_1)$ is trivial. Suppose there exist random measures B_i in $\mathcal{A}_0(1)$ and $a,b \in (0,1)$, such that

$$bB_1(\omega, t) + (1 - b)B_2(\omega, t) = aA_1(\omega, t) + (1 - a)A_2(\omega, t).$$

Since B_i can have a jump only where A_1 or A_2 has a jump, it follows that for some random variable C_i, we have $B_i = C_i A_{T_1} + (1 - C_i) A_{T_2}$; however it is straightforward to show that C_i must be $\mathcal{F}_{T_1 \wedge T_2} = \mathcal{F}_{T_1}$ - measurable, and hence constant a.s. Consequently T_1 and T_2 define an edge.

Conversely, suppose T_1 and T_2 define an edge. Then going to the random measures we have

$$\tfrac{1}{2}(A_1 + A_2) = \tfrac{1}{2}[A_1 \vee A_2 + A_1 \wedge A_2].$$

By hypothesis there then exists a constant c so that

$$A_1(\omega, t) \vee A_2(\omega, t) = cA_1(\omega, t) + (1 - c)A_2(\omega, t).$$

If there is a set with positive measure such that $T_1 < t < T_2$, then

$$1 \vee A_2(\omega, t) = 1 = c + (1 - c)A_2(\omega, t),$$

which implies the constant c equals one. If $T_2 < T_1$ is also possible on a set with positive measure, we obtain c equals zero. Both cases can't occur, and we assume $T_1 \leq T_2$.

Now suppose $\mathcal{F}(T_1)$ is not trivial. Then there exists a nonconstant $\mathcal{F}(T_1)$-measurable random variable $C(\omega)$ with $0 < C(\omega) < 1$. If we define

$$B_1(\omega, t) = \big(1 - C(\omega)\big)A_1(\omega, t) + C(\omega)A_2(\omega, t)$$

and

$$B_2(\omega, t) = C(\omega)A_1(\omega, t) + \big(1 - C(\omega)\big)A_2(\omega, t),$$

then B_1 and B_2 are both in $\mathcal{A}_0(1)$ and

$$\tfrac{1}{2}(B_1 + B_2) = \tfrac{1}{2}(A_1 + A_2).$$

However, B_1 and B_2 cannot be represented as a convex combination of A_1 and A_2 over the nonnegative reals, and that contradiction completes the proof. $\square$

4. Compactness of $A_1(1)$. In this section we deal with problems qualitatively different from the algebraic topics of Sections 2 and 3. In particular, the role of a fixed initial distribution μ means we shall be dealing with one family of probability measures on (Ω , F) :

$$(4.1) \qquad P_b^\mu(\Lambda) = \int \mu(dx) P_b^x(\Lambda) \equiv \int \mu(dx) P^x(\theta_b^{-1}\Lambda).$$

We hereafter write P for P_0^μ .

Let $\mathcal{B}$ denote the Borel sets on $[0, \infty]$ and let C denote the set of bounded $F^0 \times \mathcal{B}$ - measurable functions that are continuous in t. We recall that F^0 is separable, so that C is generated by a countable family of functions of the form $Y(\omega) \cdot f(t)$, where f is continuous. In fact we can and frequently will assume that f is chosen from the countable family $\mathcal{D} = \{\exp(-rt) : r \in Q\}$, where Q denotes the set of nonnegative rationals.

In [1] Baxter and Chacon use C to define a topology on $A_0(1)$ and prove sequential compactness of $A_0(1)$ relative to it. This requires the following kind of convergence.

(4.2) Definition. A sequence (A_n) in $A_0(1)$ converges (BC) to $A \in A_0(1)$ if for all Z in C

$$P[\int_0^\infty Z(\omega, t) A_n(\omega, dt)] \to P[\int_0^\infty Z(\omega, t) A(\omega, t)].$$

Our goal is to prove the analogous result for $A_1(1)$, but that does not seem possible without restrictions on the underlying process. Our approach is to push as far as possible without additional assumptions and then to illustrate hypotheses that allow stronger conclusions. Here is the general result.

(4.3) Theorem. Assume X is a Markov process satisfying the hypotheses of Section 2 and let (A_n) be a sequence in $A_1(1)$. Then there exist a countable set H , possibly empty, a subsequence $(A_{n'})$ and $\tilde{A} \in A_1(1)$ such that

(4.4) $$\Theta_b A_{n'} \to \Theta_b \tilde{A} \qquad \text{(BC)}$$

for all b, except possibly for b in the set H. Moreover, $\tilde{A}$ is defined using $A \in \mathcal{A}_0(1)$ having the properties: $\Theta_r A \in \mathcal{A}_0(1)$ for r in $Q' \equiv H \cup Q$, $\Theta_r A_{n'}$ converges (BC) to $\Theta_r A$ for $r \in Q'$, and if r and s are in Q' with $r < b < s$, then for all $t \geq 0$

$$\Theta_r A(\omega, t) \geq \Theta_b \tilde{A}(\omega, t) \geq \Theta_s A(\omega, t).$$

(We subsequently abbreviate this as $\Theta_r A \geq \Theta_b \tilde{A} \geq \Theta_s A$.)

It is tempting to try to combine A and $\tilde{A}$ into one random measure in $\mathcal{A}_1(1)$ satisfying (4.4) for all b. Unfortunately this does not seem possible without restrictions on the process. Before going into that point further we shall prove Theorem (4.3). In doing so we will obtain the following corollary, which illuminates the role of the set H.

(4.5) <u>Corollary</u>. Suppose $\tilde{A}$ has the property that for all nonincreasing f, $P[\int_0^\infty f(t + r)\tilde{A}(\Theta_r \omega, dt)]$ is continuous in r. Then $H = \emptyset$, and (4.4) holds for all $b \geq 0$. □

The proof of (4.3) relies heavily on certain results from Baxter and Chacon [1], which we record here. Baxter and Chacon define a functional α for $Y \in L_1(P)$ and $f \in C[0, \infty]$ by

(4.6) $$\alpha(Y, f) = P[Y(\omega) \cdot \int f(t)A(\omega, dt)]$$

and establish that functionals of the form (4.6) are characterized by the following properties

(4.7a) α is bilinear and positive : $0 \leq Y$ and $0 \leq f$ imply $0 \leq \alpha(Y,f)$;

(4.7b) $\alpha(1, 1) = 1$; $|\alpha(Y, f)| \leq \|Y\| \cdot \|f\|_\infty$;

(4.7c) for all b, supp (f) $\equiv$ support (f) $\subset$ [0, b] implies $\alpha(Y, f) = \alpha\left(P(Y|F_b), f\right)$.

(We note that to fulfill the first condition in (4.7b) we need to define $A(\omega, \{\infty\})$ as $1 - A(\omega, \infty)$.)

We need two additional conditions. Let $r \geq 0$ be fixed.

(4.7d) $\text{supp}(f) \subset [0,r]$ implies $\alpha(Y, f) = 0$;

(4.7e) $\alpha(Y, f) = \alpha\left(P(Y|G_r)\right)$, where $G_r = \theta_r^{-1}(F)$.

(4.8) Lemma. A functional α has the representation (4.6) with $A = \Theta_r B$ if and only if (4.7a-e) hold; B is unique up to P a.s. equivalence.

Proof. If $A = \Theta_r B$, it is easy to check that the conditions in (4.7) hold. Conversely, as in [1], (4.7a-c) produce A, and one need only verify it is of the asserted form. Fixing f and defining Y as $\int fA(\omega, dt)$, we can use (4.7e) to prove

$$P[\left(Y - P(Y|G_r)\right)^2] = 0$$

i.e., Y is G_r-measurable. Repeating this for a countable determining class in $C[0, \infty]$ gives existence of a random measure B such that $A(\omega, dt) = B(\theta_r\omega, dt)$ almost surely. Moreover, (4.7d) shows that support of both measures is $[r, \infty]$ a.s. Redefining B, we have $A = \Theta_r B$. The verification that B is unique up to a.s. equivalence is then immediate. (Actually, we have uniqueness with respect to the measure P_r^μ on (Ω, F).) $\square$

Another result from [1] that we shall find useful is:

(4.9). Let S and T be in $T_0(1)$, with A and B their corresponding measures. Then the following are equivalent:

(i) $S(\omega, \cdot) \leq T(\omega, \cdot)$ a.s. ;

(ii) $A(\omega, t) \geq B(\omega, t)$ a.s. ;

(iii) If f is nonincreasing on $[0, \infty]$ then
$\int f(t)A(\omega, dt) \geq \int f(t)B(\omega, dt)$ a.s.

Here is the first step in proving (4.3).

(4.10) Proposition. Given a sequence (A_n) in $\mathcal{A}_1(1)$, there exist a subsequence $(A_{n'})$ and random measures $\{B_b : b \geq 0\}$ such that:

(i) for each b, $\Theta_b A_{n'} \to \Theta_b B_b$ (BC) ;

(ii) there exists a set Ω_0 of full measure such that if $0 \leq b \leq c$, then $\Theta_b B_b \geq \Theta_c B_c$ on Ω_0 ;

(iii) except possibly for B in a countable set H

$$\lim_{c \downarrow\downarrow b} P[Y \cdot \int f(t + c)B_c(\theta_c \omega, dt)] = P[Y \cdot \int f(t + b)B_b(\theta_b \omega, dt)]$$

for f a continuous nonincreasing function and $Y \geq 0$.

Proof. Let Q denote the rationals as before and select a subsequence $A_{n'}$ such that for each rational r , $\Theta_r A_{n'}$ converges (BC) to a limit random measure A_r in $\mathcal{A}_0(1)$. Since (4.7 d, e) hold for $\Theta_r A_{n'}$, they hold for A_r and hence A_r is of the form $\Theta_r B_r$. Thus

$$\int_0^\infty f(t)A_r(\omega, dt) = \int_0^\infty f(t + r)B_r(\theta_r \omega, dt).$$

Since $A_n \in \mathcal{A}_1(1)$, it follows that for $Y \geq 0$,

$$P[Y \cdot \int_0^\infty f(t + r)A_n(\theta_r \omega, dt)] \geq P[Y \cdot \int_0^\infty f(t + s)A_n(\theta_r \omega, dt)],$$

provided f is nonincreasing and $r \leq s$. For r and s in Q , this inequality persists in the limit, and we can define a set Ω_0 of full P-measure such that (ii) holds for all b and c in Q . Moreover, since $g(r) = g(r, Y, f) \equiv$

$P[Y \int_0^\infty f(t + r)B_r(\theta_r\omega, dt)]$ is monotone on Q, we can define a countable set H such that for b not in H, g is continuous across b for all Y and f in the family $\mathcal{D}$ of functions $f(t) = \exp(-rt)$, $r \in Q$.

We now extract a further subsequence so that $\theta_r A_{n'}$ converges to $\theta_r B_r$ for $r \in Q' = H \cup Q$. Since Q' is also countable, the preceding analysis applies to the new measures, and we use the same letters. On Ω_0 we define B_b for all b by

$$\int_0^\infty f(t + b)B_b(\theta_b\omega, dt) = \sup_{\substack{s \in Q' \\ s \geq b}} \int_0^\infty f(t + s)B_s(\theta_s\omega, dt).$$

Note that the family $\{\theta_b B_b : b \geq 0\}$ extends the family with b restricted to Q' and that conditions (ii) and (iii) have been established.

For condition (i) we observe that (BC) convergence is valid by definition for b in Q'. For b not in Q' we have

$$\begin{aligned} &P[Y \cdot \int_0^\infty f(t + s)B_s(\theta_s\omega, dt)] \\ &\leq \underline{\lim}\, P[Y \cdot \int_0^\infty f(t + b)A_{n'}(\theta_b\omega, dt)] \\ &\leq \overline{\lim}\, P[Y \cdot \int_0^\infty f(t + b)A_{n'}(\theta_b\omega, dt)] \\ &\leq P[Y \cdot \int_0^\infty f(t + r)B_r(\theta_r\omega, dt)] \end{aligned}$$

provided $r < b < s$, r and s are in Q', $0 \leq Y$, and f nonincreasing. Since b is not in Q', we have continuity across b for a generating class of $\mathcal{C}$. Therefore the limit exists and can be represented using $\theta_b B_b$. This completes the proof of (4.10). $\square$

We are now able to prove Theorem (4.3).

Proof of Theorem (4.3): It would be pleasant if we could simply drop the b as a subscript of B and aver that we had obtained the limit random measure. Unfortunately, that seems to be the most difficult step. Our approach is to define a new initial measure ρ and a limit distribution relative to P^{ρ}. The properties of ρ will enable us to define $\tilde{A}$.

Let $\{g_s : s \in Q\}$ denote a countable family of nonnegative, bounded continuous functions that generates $\mathcal{B}$; then with a(s) and b(r) denoting positive normalizing constants, we define

$$\rho(dx) = \sum_{s \in Q} a(s) \int_0^{\infty} g_s(b) e^{-b} P^{\mu}(X_b \in dx)\, db + \sum_{r \in Q'} b(r) P^{\mu}(X_r \in dx).$$

Next apply Baxter and Chacon's result to $(A_{n'})$ relative to the measure P^{ρ} on $(\Omega, \mathcal{F})$ and extract yet another subsequence converging (BC) to a random measure $A \in \mathcal{A}_0(1)$. We then use standard techniques to obtain an $\mathcal{F}^0$-measurable random measure P^{ρ}-equivalent to A and denoted by the same letter.

Now it is clear that on path space P_r^{μ} is absolutely continuous with respect to P^{ρ} for $r \in Q'$. Hence there exists a Radon-Nikodym derivative W_r such that

$$\begin{aligned} P^{\mu}[Y \circ \theta_r \cdot \int f(t + r) A_{n'}(\theta_r\omega, dt)] &= P_r^{\mu}[Y \cdot \int f(t + r) A_{n'}(\omega, dt)] \\ &= P^{\rho}[Y \cdot W_r \int f(t + r) A_{n'}(\omega, dt)] \\ &\to P^{\rho}[Y \cdot W_r \int f(t + r) A(\omega, dt)] \\ &= P_r^{\mu}(Y \cdot \int f(t + r) A(\omega, dt)]. \end{aligned}$$

It follows that on a set of full P-measure we have $\Theta_r B_r = \Theta_r A$ for all r in Q'. In effect we can eliminate the subscript from B_r for all $r \in Q'$. Moreover, we can retain the inequalities $\Theta_r A \geq \Theta_s A$, $r < s \in Q'$, on this set of full measure.

A repetition of the argument using absolute continuity confirms that for each s

$$\int_0^\infty g_s(b) e^{-b} P_b^\mu [Y \cdot \int_0^\infty f(t+b) A_{n'}(\omega, dt)] db$$

converges to the same expression with A replacing $A_{n'}$. Indeed with our usual $Y \geq 0$ and $f \in \mathcal{D}$, we can get an upper bound for the expression above, to wit

$$\sum_k \int_{b_k}^{b_{k+1}} g_s(b) e^{-b} P_{b_k}^\mu [Y \cdot \int_0^\infty f(t+b_k) A_{n'}(\omega, dt)] db$$

and a lower bound of the same form but using b_{k-1} in place of b_k inside the integral. If the b_k are restricted to Q', we can pass to the limit and express the upper and lower bounds using B_{b_k} and $B_{b_{k+1}}$ as the random measures. Finally, assume the mesh size of the partition is taken smaller and use countability of the set of discontinuity points to obtain

$$\int_0^\infty g_s(b) e^{-b} P_b^\mu [Y \cdot \int_0^\infty f(t+b) B_b(\omega, dt)] db$$
$$= \int_0^\infty g_s(b) e^{-b} P_b^\mu [Y \cdot \int_0^\infty f(t+b) A(\omega, dt)] db. \tag{4.11}$$

Since A was chosen F^0-measurable, it follows that for Y F^0-measurable the function $G(\omega, b)$ defined as $Y(\omega) \cdot \int_0^\infty f(t+b) A(\omega, dt)$ if $F^0 \times \mathcal{B}$-measurable. By assumptions on the (P^x), that means $P^{X(b,\omega)}[G(\omega', b)]$ is $F^0 \times \beta$-measurable

and Fubini's theorem than gives the $\mathcal{B}$-measurability of $P^{\mu}[P^{X(b,\omega)}[G(\omega',b)]]$.

We can argue that the corresponding expression with B_b in place of A is also $\mathcal{B}$-measurable, since it is the limit of $\mathcal{B}$-measurable functions. It then follows that (4.11) holds for all Borel measurable g, as well as the g_s , from which we can conclude that for all b except those in a real null set,

$$P_b^{\mu}[Y \cdot \int_0^{\infty} f(t + b)B_b(\omega, dt)] = P_b^{\mu}[Y \cdot \int_0^{\infty} f(t + b)A(\omega, dt)].$$

Hence, except for b in a null set

$$\Theta_b B_b = \Theta_b A \quad , \quad \text{a.s. } P^{\mu}.$$

Again by Fubini's theorem, on a set of full P^{μ}-measure $\Theta_b B_b = \Theta_b A$, except for a real null set depending on ω. But this means that if we use the concept of essential limits - that is, with respect to a topology on $[0, \infty]$ in which open sets are Borel open sets minus a set of Lebesgue measure zero - we can define a random measure $\tilde{A}$ by

$$(4.12) \qquad \int_0^{\infty} f(t)\tilde{A}(\omega, dt) \equiv \operatorname*{ess\,lim}_{b\downarrow 0} \int_0^{\infty} f(t + b)A(\theta_b\omega, dt),$$

for f nonnegative, nonincreasing and continuous. Note that the monotonicity property of $\Theta_b B_b$ and its a.s. equivalence to $\Theta_b A$ also give

$$\int_0^{\infty} f(t)\tilde{A}(\omega, dt) = \lim_{b\downarrow\downarrow 0} \int_0^{\infty} f(t + b)B_b(\theta_b\omega, dt).$$

The key feature of (4.12) is that we obtain the same random measure $\tilde{A}(\omega', dt)$ for all ω' of the form $\omega' = \theta_{b_1}\omega_1 = \theta_{b_2}\omega_2$, where ω_1 and ω_2 lie in the set of full measure defined in the preceding paragraph. This is because we can use $A(\theta_b\omega', \cdot) = A(\theta_{b+b_1}\omega, \cdot) = A(\theta_{b+b_2}\omega, \cdot)$ in

the limit and need not worry about accumulation of null sets that would result were we to try to use B_b directly in (4.12).

Those unfamiliar with the concepts used in (4.12) can refer to Walsh [15], in which these topics are described and from which the F-measurability of $\tilde{A}$ follows. Since $\Theta_r B_r = \Theta_r A$ on a set of full measure, the inequalities asserted in (4.3) have been established, and at times b not in H the convergence of $\Theta_b A_{n'}$ to $\Theta_b \tilde{A}$ is immediate. We have thus devised a way to "drop the subscripts from B_b" - at least to the point of reducing the statement to two limit measures $\tilde{A}$ and A. That is enough for (4.3) and completes the proof of the theorem. □

Without additional restrictions on the process we cannot eliminate the need for two random measures in the statement of (4.3). In view of the generality in which the theorem is formulated, that is not entirely unexpected, but neither is it entirely satisfactory. The difficulty is that in regularizing A to get $\tilde{A} \in A_1(1)$, we imposed a type of right-continuity not required for $A_1(1)$. The effect is that we cannot assert that $(A_{n'})$ converges to $\tilde{A}$ even though we do know $(A_{n'})$ converges to A.

As an example, let X be uniform motion to the right on the line and let A_n correspond to the first hit of $\{1/n\} \cup [1, \infty)$. Then A evidently corresponds to the first hit of $\{0\} \cup [1, \infty)$; in particular if $X_0(\omega) = 0$, $A(\omega, \cdot)$ puts all its weight at $\{0\}$. However, for that same ω, $\tilde{A}(\omega, \cdot)$ corresponds to the first hit of $[1, \infty)$ and thus is not the limit of the A_n.

There are a variety of assumptions restricting either the process X or the sequence (A_n) that enable us to make sharper assertions. The first is motivated by the counterexample above.

(i) Uniformity of convergence of right-continuous A_n. Without loss of generality, replace the subsequence $(A_{n'})$ by the original sequence. Let Z_r stand for $Y(\omega) \cdot f(t + r)$ for our usual Y and f and use $(Z_r , \Theta_r A)$ to denote

$$P^{\mu}[\int_0^{\infty} Z_r(\omega, t)A(\theta_r\omega, dt)].$$

We make two assumptions:

(4.13a) for all sufficiently large n, $\lim_{r\to 0} (Z_r , \Theta_r A_n) = (Z , A_n)$.

(4.13b) for each such Z there is a $\delta > 0$ such that

$$\lim_{n\to\infty} \sup_{r\in Q, r<\delta} |(Z_r , \Theta_r A_n) - (Z_r , \Theta_r A)| = 0.$$

It follows from (4.13) that (A_n) converges to $\tilde{A}$. To see this fix Z and the δ guaranteed by (4.13b). Then

$$\begin{aligned}|(Z, \tilde{A}) - (Z, A)| &\le |(Z, \tilde{A}) - (Z_r , \Theta_r A)| \\ &+ |(Z_r , \Theta_r A) - (Z_r , \Theta_r A_n)| + |(Z_r , \Theta_r A_n) - (Z , A_n)| \\ &+ |(Z , A_n) - (Z , A)| .\end{aligned}$$

The fourth term on the right can be made small for large n, as can the second term uniformly in $r < \delta$, by virtue of (4.13b). Fixing such an n, we can then make the first and third terms small by choosing small r. Hence $(Z , \tilde{A}) = (Z , A)$ for a generating class of Z, which verifies the assertion.

(ii) Markov chains with countable, stable states. The conditions in (4.13) hold in this case, so that $A_1(1)$ is sequentially compact.

(iii) Discrete time Markov processes. In this case there is no difficulty with null sets accumulating, and the Baxter-Chacon argument easily gives sequential compactness of $A_1(1)$.

(iv) Domination of the transition probabilities. (We are indebted to J. Glover for suggesting this condition.)

(4.14) Proposition. Suppose η is a measure an S such that

$$P^{\mu}(X_b \in \cdot) << \eta(\cdot)$$

for all $b \geq 0$. Then $\bar{A}_1(1)$ is sequentially compact.

Proof. It is easy to use an absolute continuity argument to show that for all $b \in Q'$, $\Theta_b\tilde{A} = \Theta_b A$ almost surely. Hence $\Theta_b A$ works as the limit measure for all b and the almost sure inequalities defining $\bar{A}_1(1)$ follow immediately. ∎

5. Applications. As noted in the introduction, motivation for studying structural properties of randomized times arose from problems concerning transient Markov processes. In this section we consider three applications of the material from the preceding sections. In the first we use a modification of the remplissage (filling) scheme to give an explicit construction of a randomized terminal time that realizes (1.1), thereby solving in $T_2(1)$ Skorokhod's problem for finite state Markov chains. In the second application we pursue an analogous example for a transient Markov chain with countable state space and see how (4.13) can be applied. The third application concerns Rost's original work [12] and illustrates how the class of randomized times could be narrowed in his context.

We begin by assuming X is a Markov chain with a countable state space S. Let A be a finite subset of S and μ a fixed probability with support in A. Define by $M(\mu)$ the set of probabilities η with support in A and such that

(5.1) $$\int f d\eta \leq \int f d\mu$$

for all $f \in \mathcal{E}$, the cone of bounded excessive functions. (The integral in this context is just a finite sum, but in subsequent discussions we shall interpret (5.1) as an integral over a more general space.)

We have the following result.

(5.2) <u>Theorem</u>. For each η in $M(\mu)$ there exist an integer $N \le \text{card}\,(\text{supp}\,(\eta))$, a set of positive reals $\{s_i : 1 \le i \le N\}$ with $\sum_1^N s_i = 1$ and a strictly decreasing family $\{A_i : 1 \le i \le N\}$ of subsets of $\text{supp}(\eta)$ such that:

(a) $P^{\mu}(D_i \le \infty) = 1$, $1 \le i \le N$, where $D_i = \inf\{n > 0 : X_n \in A_i\}$, and $A_1 = \text{supp}\,(\eta)$;

(b) $\eta(x) = \int_0^1 P^{\mu}\big(X(T_u) = x\big)du$,

where T is the randomized terminal time

$$T(\omega, u) = D_i(\omega), \qquad u_{i-1} < u \le u_i ,$$

with $u_0 = 0$ and $u_i = \sum_1^i s_j$.

<u>Proof</u>. In the filling scheme cited above one fills in the measure η step by step, with each step corresponding to one time unit. In our approach each step corresponds to one hitting time. Specifically, let $\mu_0 = \mu$, $\eta_1 = \eta$ and $A_1 = \text{supp}\,(\eta_1)$. Since the function $f(x) = P^x(D_1 < \infty)$ is excessive, (5.1) gives

$$P^{\mu_0}(D_1 < \infty) = \int f_1 d\mu_0 \ge \int f_1 d\eta_1 = P^{\eta_1}(D_1 < \infty) = 1 .$$

Thus the measure $\mu_1(x) = P^{\mu_0}\big(X(D_1) = x\big)$ is a probability with $\text{supp}(\mu_1) \subset \text{supp}(\eta_1)$. If $\mu_1 = \eta_1$ we are done. Otherwise define

$$t_1 = \max\{t : \eta_1 - t\mu_1 \ge 0\}$$

$$\eta_2 = (1 - t_1)^{-1}(\eta_1 - t_1\mu_1),$$

and let $A_2 = \text{supp}(\eta_2) \subsetneq A_1$. To see that $\eta_2 \in M(\mu_1)$, suppose $f \in \mathcal{E}$ and define its réduite

$$r_1f(x) = P^x[f(X(D_1)) \; ; \; D_1 < \infty].$$

The function r_1f is excessive, and $r_1f \leq f$, with equality on A_1 . Hence

$$\int fd\eta_1 = \int r_1fd\eta_1 \leq \int r_1fd\mu_0 = \int fd\mu_1 .$$

It follows that

$$\int fd\eta_2 = (1 - t_1)^{-1}[\int fd\eta_1 - t_1\int fd\mu_0] \leq \int fd\eta_1 \leq \int fd\mu_1 ,$$

confirming $\eta_2 \in M(\mu_1)$. Setting $A_2 = \text{supp}(\eta_2)$, the function $f(x) = P^x[D_2 < \infty]$ is excessive, and

$$P^{\mu_0}(D_2 < \infty) = P^{\mu_1}(D_2 < \infty) = \int fd\mu_1 \geq \int fd\eta_2 = 1 .$$

Proceeding recursively, define $\mu_2(x) = P^{\mu_0}(X(D_2) = x)$. If $\mu_2 = \eta_2$, we have

$$\eta_1(x) = t_1P^{\mu_0}[X(D_1) = x] + (1 - t_1)P^{\mu_0}[X(D_2) = x]$$

and need only set $N = 2$, $s_1 = t_1$ and $s_2 = 1 - t_1$ to complete the proof. Otherwise, we define

$$t_2 = \max\{t : \eta_2 - t\mu_2 \geq 0\},$$

$$\eta_3 = (1 - t_2)^{-1}(\eta_2 - t_2\mu_2),$$

and set $A_3 = \text{supp}(\eta_2) \subsetneq A_2$. We then repeat the argument of the preceding paragraph.

Since the sets A_i are strictly decreasing this procedure ends after at most card $(\text{supp}(\eta))$ steps, and the constants can be identified as $s_1 = t_1$ and $s_i = \left(\prod_{1}^{i-1} (1 - t_j)\right)t_i$, $2 \leq i \leq N$, where N is the number of steps. $\square$

Remplissage with hitting times also applies in the finite state case even when (5.1) fails. One proceeds in the same fashion, but since $D_i = \infty$ is possible, the μ_i may be sub-probability measures, the η_i may have mass greater than 1 and the t_i must be constrained not to exceed one - all of this in order to account for mass that "escapes."

(5.3) Theorem. Let μ and η be probability measures on the finite set A and assume A is transient for the process. Then there exist an integer $N \geq 0$, strictly decreasing subsets $\{A_i : 1 \leq i \leq N+1\}$ and positive constants s_i , $1 \leq i \leq N$ such that $\Sigma s_i = 1$ and

$$\eta(x) = \sum_{i=1}^{N} s_i P^{\mu}[X(D_i) = x] + (\eta - \mu)H_{N+1}(x) ,$$

where $H_{N+1}(x, y) = P^x[X(D_{N+1}) = y , D_{N+1} < \infty]$.

Proof. One proceeds as in (5.2), stopping at stage N if either $\mu_N = \eta_N$ or $t_N \geq 1$. In the first case we have (5.2). In the second case, the s_i are defined as before and

$$\eta = \sum_{i=1}^{N} s_i\mu_i + \eta_\infty ,$$

where

$$\eta_\infty = \Big(\prod_{1}^{N-1} (1 - t_j)\Big)(\eta_N - \mu_N) .$$

If $A_{N+1} = \mathrm{supp}(\eta_N - \mu_N)$, the proof will be completed by showing η_∞ has the asserted form. For $f \in E$

$$\int f d\eta_\infty = \int f d\eta - \sum_{i=1}^{N-1} s_i \int f d\mu_i - s_N \int f d\mu_N .$$

By construction $\eta = \sum_{1}^{N-1} s_i\mu_i$ on $A_1 - A_N$,

$$\int f d\eta_\infty = \int_{A_N} f d\eta - \sum_{i=1}^{N-1} s_i \int_{A_N} f d\mu_i - \int f d\mu_N + (1 - s_N)\int f d\mu_N .$$

Since f equals its réduite $r_N f$ on A_N , we can replace f by $r_N f$ above to obtain

$$\int f d\eta_\infty = \int_{A_N} r_N f d(\eta - \mu) - \sum_{i=1}^{N-1} s_i \int_{A_N} r_N f d\mu_i + (1 - s_N)\int f d\mu_N .$$

(Recall that $\int f d\mu_N = \int r_N f d\mu$.) The integrals in the summation have the form $\int f d\mu_N$, yielding the last step:

$$\int f d\eta_\infty = \int r_N f d(\eta - \mu) = \int f d(\eta_N - \mu_N).$$

It follows that η_∞ and $\eta_N - \mu_N$ have the same potential and are thus equal. It is then easy to show that $\eta_N - \mu_N = \eta_{N+1} - \mu_{N+1}$, where $A_{N+1} = \mathrm{supp}(\eta_\infty)$, and this completes the proof. $\square$

Note that the procedure always terminates in a finite number of steps. This contrasts with the time-step method of remplissage, which may require infinitely many steps. However, it is easy to construct examples which show the approach used here is generally applicable only in the finite state case. In fact these examples motivate Theorem (4.3), which would allow us to use a limiting process to define an appropriate randomized quasi-terminal time, at least in the Markov chain case.

Thus, for our second application we assume X is a transient Markov chain on a countable state space, μ and η are measures satisfying (5.1) on this countable set, and we replace the assumption of finite support for μ and η by the assumption that $\mathrm{supp}(\eta)$ is a transient set. Under these hypotheses, (5.1) is equivalent to $\mu G(x) \geq \eta G(x)$ for all $x \in S$, where G is the potential matrix.

Define $T(C)$ as $\inf\{n \geq 1 : X_n \in C\}$, $D(C)$ as $\inf\{n \geq 0 : X_n \in C\}$, $T_\eta = T(\mathrm{supp}(\eta))$, and $D_\eta = D(\mathrm{supp}(\eta))$. Then $B_0 = \{x : P^x(T_\eta < \infty) < 1\}$ cannot be empty. In fact, since $P^\mu[D_\eta < \infty] = 1$ by (5.1), we have a more precise result.

(5.4) Lemma. If (5.1) holds, then $P^\mu(D(B_0) < \infty) = P^\eta(D(B_0) < \infty) = 1$. $\square$

We then have the following.

(5.5) Theorem. Suppose μ and η satisfy (5.1) and $\mathrm{supp}(\eta)$ is transient. Then there exists a randomized quasi-terminal time T such that for all $x \in S$

$$\eta(x) = \int_0^1 P^\mu(X(T_u) = x)du .$$

Proof. Let C_1 be a finite subset of $\mathrm{supp}(\eta) - B_0$ and let $A_1 = C_1 \cup B_0$. Using the réduite as before it is easy to show that μ_1 dominates η_1 in the potential ordering, where μ_1 and η_1 are the balayages of μ and η onto A_1 . By (5.4), $D(A_1) < \infty$ a.s. P^μ and P^η , so that μ_1 and η_1 are probabilities.

We will now show that the recursive procedure in Theorem (5.2) works in this context as well. In fact, inspection of that procedure shows we need only verify $\mu_1(x) \leq \eta_1(x)$ for all $x \in B_0$. That being so, the set B_0 will be carried intact in the iteration, and the same proof applies. Define, then,

$$G_1 = \{x \in B_0 , \mu_1(x) > \eta_1(x)\},$$

$$G_2 = \{x \in B_0 , \mu_1(x) \leq \eta_1(x)\},$$

and

$$f(x) = P^x(D(C_1 \cup G_2) < \infty).$$

Then f is excessive, $\int f d\eta_1 \leq \int f d\mu_1$, and from that we can obtain

$$\int_{G_1} (1 - f) d\mu_1 \leq \int_{G_1} (1 - f) d\eta_1 .$$

Since $\mu_1(x) > \eta_1(x)$ on G_1, on G_1 we have

$$f(x) = P^x\left(T(C_1 \cup G_2) < \infty\right) = 1 .$$

But for all $x \in B_0$, $P^x(T_{\eta_1} < \infty) < 1$ by definition, which leads us to conclude that G_1 is empty, and $\mu_1(x) \leq \eta_1(x)$ everywhere on B_0.

We now proceed as follows. Let $(C_n : n \geq 1)$ be an increasing sequence of sets converging to $\mathrm{supp}(\eta) - B_0$, let η_n be the balayage of η onto $C_n \cup B_0$, and let T_n be the randomized terminal times constructed in Theorem (5.2):

$$P^{\mu}_{T_n}(x) \equiv \int_0^1 P^{\mu}[X\left(T_n(u)\right) = x]du \tag{5.6}$$

$$= \int_0^1 P^{\mu_n}[X\left(T_n(u) = x\right)]du = \eta_n(x) ,$$

where μ_n is the balayage of μ onto $C_n \cup B_0$. Condition (iii) at the end of Section 4 applies, so we can extract a subsequence converging (BC) to a randomized quasi-terminal time T such that $\eta G(x) \geq P^{\mu}_{T} G(x) \geq \eta_n G(x)$ for all n. Since $\eta_n(y) \geq \eta(y)$ on $C_n \cup B_0$, it follows that $\eta G = P^{\mu}_{T} G$, and that forces η to equal P^{μ}_{T}. □

We have relied upon the assumption that time and state space are countable, and it is reasonable to ask whether this assumption is indispensable. In fact it is not, and we shall indicate how Rost's original proof remains

valid with times restricted to $\tilde{\mathcal{T}}_1(1)$, at least under the assumption of (4.14). The first thing to record is Meyer's observation that Baxter and Chacon's work can be extended. For the details we refer the reader to [11] and content ourselves here with stating the relevant result.

(5.7) Theorem. Suppose X is a standard, transient Markov process with potential kernel G. Then if (A_n) converges (BC) to A, for any bounded $\mathcal{S}$-measurable function f

$$P^{\mu}[\int_0^{\infty} Gf(X_t)A_n(\omega, dt)] \to P^{\mu}[\int_0^{\infty} Gf(X_t)A(\omega, dt)]. \quad \square$$

Now suppose X is a standard, transient Markov process on $(S , \mathcal{S})$ and that μ and η satisfy (1.1) in that context. Rost [12] approaches the problem of finding an S in $\mathcal{T}_0(1)$ such that $\eta = P^{\mu}_S$ by first obtaining a "maximal" S such that P^{μ}_S dominates η in the potential ordering. That argument can be carried over to $\tilde{\mathcal{T}}_1(1)$ by using sequential compactness, at least in the context of Proposition (4.14). The next step is to use the potentials $P^{\mu}_S G$ and ηG to construct a certain nonrandomized, terminal time T that is conjoined with S to form $S' = \Theta_S T$. With the sort of algebraic structure introduced in Section 3, Rost shows a convex combination of S and S' also dominates η and thus by maximality $P^{\mu}_S[T = 0] = 1$. The form of T coupled with an extension of Hunt's domination principle forces equality of $P^{\mu}_S G$ and ηG , *completing the proof.*

If one could show S' was also in $\tilde{\mathcal{T}}_1(1)$, then the same arguments Rost used would apply, and we could assert that there is an S in $\tilde{\mathcal{T}}_1(1)$ such that $\eta = P^{\mu}_S$, at least in the context of (4.14). We conclude by stating the missing fact as a lemma, omitting the proof, which is merely a matter of tracing out the definition.

5.8) <u>Lemma</u>. Suppose $T \in \mathcal{T}_1(1)$ and $S \in \tilde{\mathcal{T}}_1(1)$. Then $S' = S + T \circ \theta_S$ is in $\tilde{\mathcal{T}}_1(1)$.

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